Assignment 2.

This homework is due *Thursday*, September 12.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Quick reminder

A relation $R \subseteq X \times X$ is an *equivalence* relation if it is

- reflexive: $\forall x \in X \; xRx$,
- symmetric: $\forall x, x' \in X$ if xRx' then $x'Rx$,
- transitive: $\forall x, x', x'' \in X$ if xRx' and $x'Rx''$ then xRx''.

A relation $R \subseteq X \times X$ is a *partial ordering* (partial order) if it is

- reflexive: $\forall x \in X \; xRx$,
- antisymmetric: $\forall x, x' \in X$ if xRx' and $x'Rx$ then $x = x'$,
- transitive: $\forall x, x', x'' \in X$ if xRx' and $x'Rx''$ then xRx''.

A partial order is called *total* if $\forall x, x' \in X \; xRx'$ or $x'Rx$.

2. Exercises

(1) Determine whether the following are equivalence relations on X:
   (a) $X = \mathbb{R}$, $x \approx y$ if and only if $|x - y| < 0.1$.
   (b) $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, $(n_1, m_1) \equiv (n_2, m_2)$ if and only if $n_1 m_2 = m_1 n_2$. (What is a good name for this? There are at least two answers.)
   (c) $X = \{f \mid f : A \to \mathbb{R}, A \subseteq \mathbb{R}\}$ (the set of functions from a subset of $\mathbb{R}$ to $\mathbb{R}$),
   functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ "partially agree" if and only if their restrictions on $A \cap B$ are equal:
   $$f \sim g \quad \text{iff} \quad f|_{A\cap B} = g|_{A\cap B}.$$

(2) Describe all relations that are equivalences and total orderings at the same time.

(3) Determine whether the following are partial orders on X:
   (a) $X = \mathbb{R}_{>0}$ (positive reals), $x \ll y$ if and only if $y/x > 10$.
   (b) $X = \{[a, b] \mid a, b \in \mathbb{R}, a \le b\}$ (closed intervals), $[a, b] \preceq [c, d]$ if and only if $a \le c$ and $b \le d$.
   (c) $X = \{f \mid f : A \to \mathbb{R}, A \subseteq \mathbb{R}\}$. For functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, put $f \preceq g$ if and only if
   $A \subseteq B$ and $f = g|_A$; in other words, if and only if f is a restriction of g.

(4) (1.4.27) Is the set of rational numbers open? Is it closed?

— see next page —

(5) (1.4.28) Prove that $\emptyset$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed. (*Hint:* There are many ways to approach this. One of them is to use classification of open sets in $\mathbb{R}$.)
Comment. Those familiar with general topology may see that this problem asks to prove that $\mathbb{R}$ is connected.

(6) (1.4.37) Show that every open set in $\mathbb{R}$ can be represented as a countable union of closed sets.

3. Extra exercises

Problems below will only go to the numerator of your grade for this homework. Also, the due date on these problems is December, 6. That is, you can submit any of these problems any time before classes end.

(7) Give an example of a family $\mathcal{F}$ of distinct subsets of a countable set s.t. the following two conditions hold:
- $\mathcal{F}$ is uncountable.
- $\mathcal{F}$ is a *chain* with respect to set inclusion, i.e. for every two subsets A, B in the family $\mathcal{F}$, either $A \subseteq B$ or $B \subseteq A$.

(8) Give an example of a family $\mathcal{F}$ of distinct subsets of a countable set s.t. the following two conditions hold:
- $\mathcal{F}$ is uncountable.
- For any $A, B \in \mathcal{F}$, the intersection $A \cap B$ is *finite.*

(9) Suppose X and Y are two sets. Prove that if there is an injection $f : X \to Y$, and an injection $g : Y \to X$, then there is a bijection $\varphi : X \to Y$.
Comment. This problem essentially asks to prove that "injects into" is a partial order on classes of equipotence, so that it makes sense not only to say "these two sets are not of the same cardinality", but also "this set is of higher cardinality than that one".